

Universal description of three two-component fermions

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Abstract

A quantum mechanical three-body problem for two identical fermions of mass m and a distinct particle of mass m_1 in the universal limit of zero-range two-body interaction is studied. For the unambiguous formulation of the problem in the interval $\mu_r < m/m_1 \leq \mu_c$ ($\mu_r \approx 8.619$ and $\mu_c \approx 13.607$) an additional parameter b determining the wave function near the triple-collision point is introduced; thus, a one-parameter family of self-adjoint Hamiltonians is defined. The dependence of the bound-state energies on m/m_1 and b in the sector of angular momentum and parity $L^P = 1^-$ is calculated and analysed with the aid of a simple model.

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Low-energy dynamics of few two-species particles has attracted much attention as a basic quantum problem that is closely related to the investigations of ultra-cold binary quantum gases [1–7]. The principal problem is the investigation of few two-species fermions, in particular, the present Letter is aimed to study two identical fermions of mass m interacting with a distinct particle of mass m_1 . Since the few-body properties become independent of the particular form of the short-range two-body interaction in the low-energy limit, the universal description is obtained by using the contact or zero-range interaction defined by a single parameter, the two-body scattering length a . As a consequence, one expects that for the properly chosen units the few-body properties depend on a single non-trivial parameter, the mass ratio m/m_1 .

Significant advance was made in [8], where it was demonstrated that for $m/m_1 > \mu_c$ ($\mu_c \approx 13.607$), similarly to the three-boson case, the problem of three two-species fermions is ambiguously defined in the limit of zero-range interaction. For the correct formulation, an additional parameter is needed to define the oscillating wave function near the triple-collision point. By setting this parameter, one comes to the Efimov spectrum, which contains an infinite number of bound states whose binding energies tend to infinity and the ratio of subsequent energies tends to a constant.

For $m/m_1 \leq \mu_c$, one of the important results was the analytic zero-energy solution, which reveals the two-hump structure in the low-energy three-body recombination rate dependence on m/m_1 [9]. The three-body energy spectrum and the scattering cross sections for $L^P = 1^-$ were studied in [10], where two bound states were disclosed for m/m_1 increasing to μ_c . The conclusions of [10] were confirmed in [11, 12] by solving the momentum-space integral equations. The formation of the three-body clusters should affect the properties of fermionic mixtures, in particular, it indicates effective attraction between a diatomic molecule and a light particle in the p -wave state, which persists even if the three-body system is unbound. In this respect, a role of the p -wave $2+1$ scattering was discussed in [4–6, 13] and the molecule-atom p -wave attraction in ^{40}K – ^6Li mixture was detected in [7]. Furthermore, the dynamics of the ultra-cold gas consisting of three-body clusters was investigated [14, 15]. Another application to the many-body dynamics was the calculation of the third virial coefficient in the unitary limit $a \rightarrow \infty$ [16, 17].

In spite of progress, it is still necessary to correctly formulate the three-body problem for two-species fermions with zero-range two-body interaction in the mass-ratio interval

$m/m_1 \leq \mu_c$, as indicated in both physical [18–20] and mathematical [21–24] works. In this respect, the basic question is the unambiguous definition of the wave function in the vicinity of the triple-collision point. In this Letter, an additional three-body parameter b is introduced to formulate the three-body problem for $\mu_r < m/m_1 \leq \mu_c$ ($\mu_r \approx 8.619$) that corresponds to the construction of a one-parameter family of self-adjoint Hamiltonians. Within the framework of this formulation, comprehensive analytic and numerical study of the three-body bound states is performed. Due to the permutational symmetry of fermions, the states of unit total angular momentum L and negative parity P are of most interest at low energy; for this reason, the $L^P = 1^-$ sector is considered in this Letter.

The Hamiltonian in the centre-of-mass frame is the six-dimensional kinetic-energy operator $H_0 = -\Delta_{\mathbf{x}} - \Delta_{\mathbf{y}}$, where \mathbf{x} and \mathbf{y} are the scaled Jacobi coordinates and the units $\hbar = 2m/(1 + m/m_1) = 1$ are used. The two-body interaction is defined by the boundary condition for the wave function Ψ imposed on two hyper-planes corresponding to the zero distance r between either fermion and a distinct particle, $\lim_{r \rightarrow 0} \frac{\partial \ln(r\Psi)}{\partial r} = -\frac{1}{a}$. As the wave function is antisymmetric under permutation of fermions, a single condition in one pair of interacting particles is needed [10].

The formal construction of the Hamiltonian does not obviously provide an unambiguous definition of the three-body problem; in particular, one should inspect the solution at the intersection of hyper-planes (triple-collision point). To analyse the wave function, correctly define the three-body problem, and calculate the binding energies, it is suitable to expand the wave function $\Psi = \rho^{-5/2} \sum_{n=1}^{\infty} f_n(\rho) \Phi_n(\rho, \Omega)$ into a set of eigenfunctions $\Phi_n(\rho, \Omega)$ of the auxiliary problem on a hyper-sphere at fixed ρ , where $\rho = \sqrt{x^2 + y^2}$ is a hyper-radius and Ω denotes a set of hyper-angular variables [10]. This leads to an infinite set of coupled hyper-radial equations (HREs),

$$\left[\frac{d^2}{d\rho^2} - \frac{\gamma_n^2(\rho) - 1/4}{\rho^2} + E \right] f_n(\rho) - \sum_{m=1}^{\infty} \left[P_{nm}(\rho) - Q_{nm}(\rho) \frac{d}{d\rho} - \frac{d}{d\rho} Q_{nm}(\rho) \right] f_m(\rho) = 0, \quad (1)$$

where the eigenvalues of the auxiliary problem $\gamma_n^2(\rho)$ are different branches of the multi-valued function defined for $L^P = 1^-$ by

$$\frac{\rho}{a} \cos \gamma \frac{\pi}{2} = \frac{1 - \gamma^2}{\gamma} \sin \gamma \frac{\pi}{2} - 2 \frac{\cos \omega \gamma}{\sin 2\omega} + \frac{\sin \omega \gamma}{\gamma \sin^2 \omega} \quad (2)$$

and the notation $\sin \omega = 1/(1 + m_1/m)$ is used. The coupling terms $Q_{nm}(\rho)$ and $P_{nm}(\rho)$ are expressed in the analytical form via $\gamma_n^2(\rho)$ and their derivatives [10, 25, 26].

Since both eigenfunctions $\Phi_n(\rho, \Omega)$ of the auxiliary problem and the coupling terms $Q_{nm}(\rho)$ and $P_{nm}(\rho)$ are regular, the wave function Ψ for $\rho \rightarrow 0$ is basically determined by one of the channel functions $f_n(\rho)$, which corresponds to the least singular term $(\gamma_n^2 - 1/4)/\rho^2$ in the system of HREs (1), i. e., to the smallest γ_n^2 . For the sake of brevity, the channel index denoting the smallest eigenvalue, $\gamma^2(\rho)$, and the corresponding channel function, $f(\rho)$, will be omitted. To determine the channel function $f(\rho)$ up to the leading-order terms for $\rho \rightarrow 0$, one should retain in HRE the singular part $(\gamma^2 - 1/4)/\rho^2 + q/\rho$, where the notations $\gamma \equiv \gamma(0)$ and $q = \left[\frac{d\gamma^2(\rho)}{d\rho} \right]_{\rho=0}$ are introduced for brevity [27]. Generally, $f(\rho) = C_+ \varphi_+(\rho) + C_- \varphi_-(\rho)$ is a linear combination of two independent solutions, which up to the leading-order terms for $\rho \rightarrow 0$ are given by $\varphi_{\pm}(\rho) = \rho^{1/2 \pm \gamma} \left(1 + \frac{q\rho}{1 \pm 2\gamma} \right)$, except $\gamma = 0, 1/2$ when the expressions for $\varphi_{\pm}(\rho)$ contain logarithmic terms.

Consider firstly $\gamma^2 \geq 1$ ($m/m_1 \leq \mu_r \approx 8.619$) [27], in which case $\varphi_-(\rho)$ is not square-integrable when $\rho \rightarrow 0$ and should be excluded, i. e., $C_- = 0$. Thus, one should satisfy the simple condition $f(\rho) \xrightarrow{\rho \rightarrow 0} 0$, in other words, the requirement of square integrability of Ψ is sufficient. Conversely, if $\gamma^2 < 1$ ($m/m_1 > \mu_r$), both $\varphi_+(\rho)$ and $\varphi_-(\rho)$ are square-integrable and an additional boundary condition is needed if $\rho \rightarrow 0$. One should further distinguish the case $\gamma^2 < 0$ ($m/m_1 > \mu_c \approx 13.607$) [27], then $\varphi_{\pm}(\rho)$ oscillate and a standard method to lift ambiguity of the solution is to specify the constant C_-/C_+ , which must satisfy $|C_-/C_+| = 1$ to provide self-adjointness of the Hamiltonian. Thus, one comes to the family of Hamiltonians depending on a single parameter (the phase of C_-/C_+) with the well-known Efimov spectrum of bound states [8].

The aim of this Letter is the unambiguous formulation of the problem for $1 > \gamma^2 \geq 0$ ($\mu_r < m/m_1 \leq \mu_c$), which requires one defining the boundary condition for $\rho \rightarrow 0$. Again, a standard method is to specify C_-/C_+ , which should be real-valued to provide self-adjointness of the Hamiltonian. It is convenient to define the length $-\infty < b < \infty$ by $-C_-/C_+ = \pm |b|^{2\gamma} \equiv b|b|^{2\gamma-1}$, i. e., \pm refers to the sign of b . The boundary condition is straightforwardly written as

$$f(\rho) \xrightarrow{\rho \rightarrow 0} \rho^{1/2+\gamma} \mp |b|^{2\gamma} \rho^{1/2-\gamma} [1 + q\rho/(1 - 2\gamma)] \quad (3)$$

except for $\gamma = 1/2$ ($m/m_1 = \mu_e \approx 12.313$ [27]). The last term $\sim q$ can be optionally omitted if $1/2 > \gamma > 0$ ($\mu_e < m/m_1 < \mu_c$) and should be retained if $1 > \gamma > 1/2$ ($\mu_r < m/m_1 < \mu_e$), when it exceeds the first term $\rho^{1/2+\gamma}$. If $\gamma = 0$ ($m/m_1 = \mu_c$), one finds

the boundary condition either from Eq. (3) in the limit $\gamma \rightarrow 0$ or directly from $\varphi_+ \sim \sqrt{\rho}$ and $\varphi_- \sim \sqrt{\rho} \log(\rho)$,

$$f(\rho) \xrightarrow{\rho \rightarrow 0} \rho^{1/2} \log(\rho/b) , \quad (4)$$

where only $b > 0$ is allowed. In the specific case of $\gamma = 1/2$ ($m/m_1 = \mu_e$) one can take $\varphi_+ \sim \rho$ and $\varphi_- \sim 1 + q\rho \log \rho$, which gives the boundary condition

$$f(\rho) \xrightarrow{\rho \rightarrow 0} \rho - b(1 + q\rho \log \rho) . \quad (5)$$

As all other channel functions $f_n(\rho)$ tend to zero faster than $f(\rho)$ at $\rho \rightarrow 0$, it is sufficient to impose the conditions $f_n(0) = 0$ for complete formulation. For rigorous formulation the boundary condition should be imposed on the wave function Ψ , in particular, for $\mu_e < m/m_1 < \mu_c$ ($1/2 > \gamma > 0$) it follows from (3) that

$$\lim_{\rho \rightarrow 0} \left(\rho^{1-2\gamma} \frac{d \log(\rho^{2+\gamma} \Psi)}{d\rho} \pm \frac{2\gamma}{|b|^{2\gamma}} \right) = 0 ; \quad (6)$$

however, for $\mu_r < m/m_1 < \mu_e$ ($1 > \gamma > 1/2$) the boundary condition for Ψ becomes cumbersome. In addition, the boundary condition is conveniently written in terms of the channel function $f(\rho)$ and its derivative [27]. One should emphasise that the boundary condition (5) does not follow from (3) in the limit $\gamma \rightarrow 1/2$ and there is no continuous correspondence of the parameter b defined for $\gamma = 1/2$ by Eq. (5) and that defined by Eq. (3). It is suitable to consider separately the dependence on b for $m/m_1 = \mu_e$ ($\gamma = 1/2$) [27].

The boundary condition imposed when $\rho \rightarrow 0$ is equivalent to including a zero-range three-body potential, while b admits an interpretation as the generalised scattering length. This potential represents either the effect of intersection of the two-body potentials or the true three-body force. This interpretation can be illustrated by the connection of b with the parameters of a particular potential, whose range is allowed to shrink to zero [27].

The solution is simple in the limit $|a| \rightarrow \infty$ due to decoupling of HREs (1), since the eigenvalues of Eq. (2) are constants $\gamma_n^2(0)$ independent of ρ and the coupling terms $Q_{nm}(\rho)$ and $P_{nm}(\rho)$ vanish. Picking out one HRE with the smallest $\gamma_n^2(0) \equiv \gamma^2$ from the uncoupled system of HREs (1) one finds for $b > 0$ that there is one bound state whose energy $E = -4b^{-2} [-\Gamma(\gamma)/\Gamma(-\gamma)]^{1/\gamma}$ and eigenfunction $f(\rho) = \rho^{1/2} K_\gamma(\sqrt{-E}\rho)$ are expressed in terms of the gamma function and the modified Bessel function. In the limit $b \rightarrow \infty$, the bound state goes to the threshold, where it turns to the virtual state, which persists for $b < 0$ and whose energy is given by the above expression. The above expressions for $|a| \rightarrow \infty$

are a good approximation for the properties of the deep state, which exists for $|a|/b \gg 1$. Note also that redefinition of the parameter $\tilde{b} = \frac{b}{2} [-\Gamma(-\gamma)/\Gamma(\gamma)]^{\frac{1}{2\gamma}}$ gives the usual relation, $E = -\tilde{b}^{-2}$, between the energy and the scattering length.

To elucidate the qualitative features of the problem in connection with the three-body boundary condition, one constructs a simple model that provides reliable dependence of the bound-state energy on b and m/m_1 . The model is based on splitting the Hamiltonian into two parts: the singular one containing terms singular as $\rho \rightarrow 0$ and the remaining one describing a smooth dependence on m/m_1 . The former part is defined as one HRE of (1) containing the smallest $\gamma_n^2(\rho)$, moreover, only singular terms $(\gamma^2 - 1/4)/\rho^2 + q/\rho$ are retained, which allows one to obtain the correct behaviour of the solution for $\rho \rightarrow 0$ and to reproduce the attraction for finite ρ . The remaining part is defined simply as a constant $\epsilon(m/m_1)$. Explicitly, one comes to the equation $\left(\frac{d^2}{d\rho^2} - \frac{\gamma^2 - 1/4}{\rho^2} - \frac{q}{\rho} + E - \epsilon \right) f(\rho) = 0$, whose square-integrable solution is written as $f(\rho) = \rho^{1/2+\gamma} e^{-\kappa\rho} \Psi(1/2 + \gamma + q/(2\kappa), 1 + 2\gamma; 2\kappa\rho)$, where $\kappa = \sqrt{\epsilon - E}$ and $\Psi(a, c; z)$ is the confluent hyper-geometric function decaying as $z \rightarrow \infty$. The eigenenergy equation

$$(2\kappa|b|)^{2\gamma} = \mp \frac{\Gamma(2\gamma)\Gamma(1/2 - \gamma + q/(2\kappa))}{\Gamma(-2\gamma)\Gamma(1/2 + \gamma + q/(2\kappa))} \quad (7)$$

follows from boundary condition (3) for all $0 < \gamma < 1$ ($\mu_c > m/m_1 > \mu_r$) except $\gamma = 1/2$ ($m/m_1 = \mu_e$). The eigenenergy equation for $\gamma = 0$ is obtained either by taking the limit in Eq (7) or from the boundary condition (4) that gives $\log(2\kappa b) + \psi\left(\frac{1}{2} + \frac{q}{2\kappa}\right) + 2\gamma_C = 0$ for $b > 0$. Hereafter, $\psi(x)$ is the digamma function and $\gamma_C \approx 0.5772$ is the Euler–Mascheroni constant. In the special case of $\gamma = 1/2$ ($m/m_1 = \mu_e$) the eigenenergy equation $\frac{1}{q} \left(\frac{1}{b} - \kappa \right) - \log\left(\frac{|q|}{2\kappa}\right) + \psi\left(1 + \frac{q}{2\kappa}\right) + 2\gamma_C - 1 = 0$ comes from (5).

The simple model is equivalent to the generalised Coulomb problem incorporating the zero-range interaction. As follows from Eq. (7), the bound-state energies monotonically increase with increasing b ; moreover, one bound state appears if b passes through zero. It is helpful to examine two limiting cases of $b = 0$ and $b \rightarrow \infty$, which gives the eigenvalues $\kappa_{nb} = -\frac{q}{2(n+s_b\gamma)+1}$, where n is a non-negative integer and $s_0 = +1$ ($s_\infty = -1$). The bound-state energies are

$$E_{nb} = -\frac{q^2}{[2(n + s_b\gamma) + 1]^2} + \epsilon, \quad (8)$$

where n is restricted by the condition $2(n + s_b\gamma) + 1 > 0$ if $a > 0$ ($q < 0$) and $2(n + s_b\gamma) + 1 < 0$ if $a < 0$ ($q > 0$). Hereafter it is convenient to take $|a|$ as a length unit that sets the two-body

binding energy to unity. Estimating the constant $\epsilon \approx -0.5$, one finds that for $a > 0$ there are two branches below the threshold (at $E \leq -1$) if $b = 0$ and three branches if $b \rightarrow \infty$, while for $a < 0$ there is one branch below the threshold (at $E \leq 0$) if $b \rightarrow \infty$ (see Fig. 1). For $a > 0$, from Eq. (8) follows degeneracy of the branches E_{n0} and $E_{n\infty}$ ($n = 0, 1$) as $m/m_1 \rightarrow \mu_c$ ($\gamma \rightarrow 0$), E_{00} and $E_{1\infty}$ as $m/m_1 \rightarrow \mu_e$ ($\gamma \rightarrow 1/2$), and E_{00} and $E_{2\infty}$ as $m/m_1 \rightarrow \mu_r$ ($\gamma \rightarrow 1$). Moreover, from Eq. (7) it follows that as $m/m_1 \rightarrow \mu_c$ ($\gamma \rightarrow 0$) the energies for any $b < 0$ converge to either $E_{00} = E_{0\infty}$ or $E_{10} = E_{1\infty}$. As $m/m_1 \rightarrow \mu_e$ ($\gamma \rightarrow 1/2$) the energies converge to either of three options, the threshold $E = -1$, $E_{00} = E_{1\infty}$, and $-\infty$. And as $m/m_1 \rightarrow \mu_r$ ($\gamma \rightarrow 1$) the energies converge to either $E_{1\infty}$ or $E_{00} = E_{2\infty}$ as shown in Fig. 1. For $a < 0$, the energies converge to $E_{0\infty}$ as $m/m_1 \rightarrow \mu_r$ ($\gamma \rightarrow 1$) and to $-\infty$ as $m/m_1 \rightarrow \mu_e$ ($\gamma \rightarrow 1/2$) for $b \neq 0$.

The three-body bound-state energies are determined by numerical solution of the truncated system of HRE (1) complemented by boundary conditions (3), (4), and (5). The numerical method is the same as in [10, 28] apart from implementation of the boundary conditions at sufficiently small ρ . Sufficient accuracy of the calculated three-body bound-state energies is achieved by solving up to eight HREs; the results are plotted in Fig. 1. The calculated dependences are consistent with the overall predictions of the simple model.

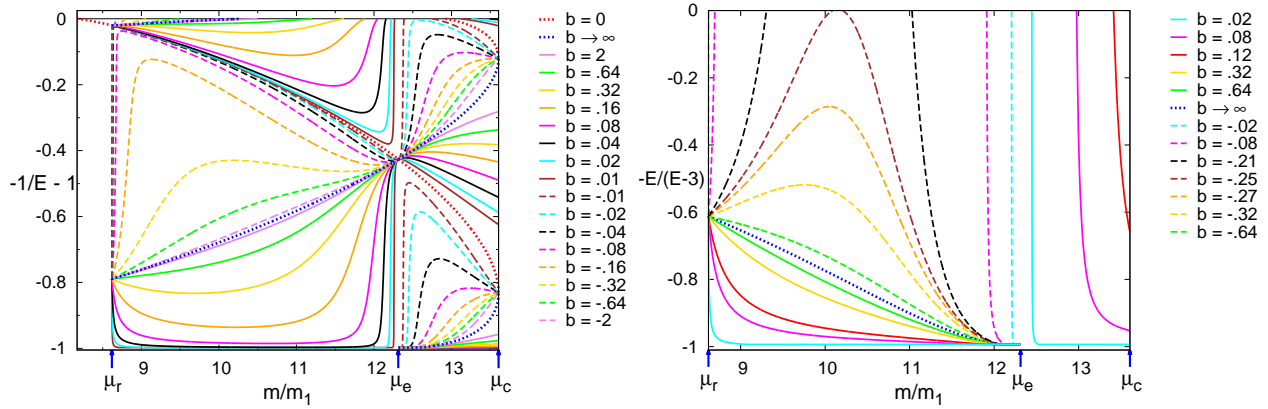


FIG. 1. Bound-state energies E as a function of m/m_1 and b for the two-body scattering length $a > 0$ (left) and $a < 0$ (right) and the energy axis scaled to map $-\infty < E < -1$ (left) and $-\infty < E < 0$ (right) to the interval $(-1, 0)$. Values μ_r , μ_e and μ_c correspond to $\gamma = 1, 1/2$ and 0 .

The energy dependence on b for fixed m/m_1 is typical of a sum of the finite-range and zero-range potentials, in particular, variation of the parameter b leads to the appearance or

disappearance of one bound state.

The calculations for $a > 0$ show that if $m/m_1 \rightarrow \mu_r$ the energies for any b converge either to $E_{1\infty} \sim -4.7477$ or to $E_{00} = E_{2\infty} \sim -1.02090$, if $m/m_1 \rightarrow \mu_e$ there is one limit $E_{00} = E_{1\infty} \sim -1.74397$, and if $m/m_1 \rightarrow \mu_c$ the energies for any $b \leq 0$ converge either to $E_{00} = E_{0\infty} \rightarrow -5.89543$ or to $E_{10} = E_{1\infty} \rightarrow -1.13767$. In agreement with [10] it is found that if $m/m_1 \leq \mu_r$, where only $b = 0$ is allowed, there is one bound state, which arises at $m/m_1 \approx 8.17259$ and naturally continues the branch E_{00} . The calculations for $a < 0$ show that if $m/m_1 \rightarrow \mu_r$ the energies for any b converge to the limit $E_{0\infty} \rightarrow -4.7147$. If $m/m_1 \rightarrow \mu_r$, the limit $E_{0\infty}$ for $a < 0$ coincides with the limit $E_{1\infty}$ for $a > 0$, as predicted by the simple model (8).

Elaborate calculations were carried out to determine the critical parameter $b_c(m/m_1)$, for which the bound-state energy coincides with the threshold [27]. The lines $b_c(m/m_1)$, $b = 0$, and $m/m_1 = \mu_e$ form boundaries of the domains of the definite number of bound states in the $m/m_1 - b$ plane as presented in Fig. 2. Few points of the dependence $b_c(m/m_1)$ are

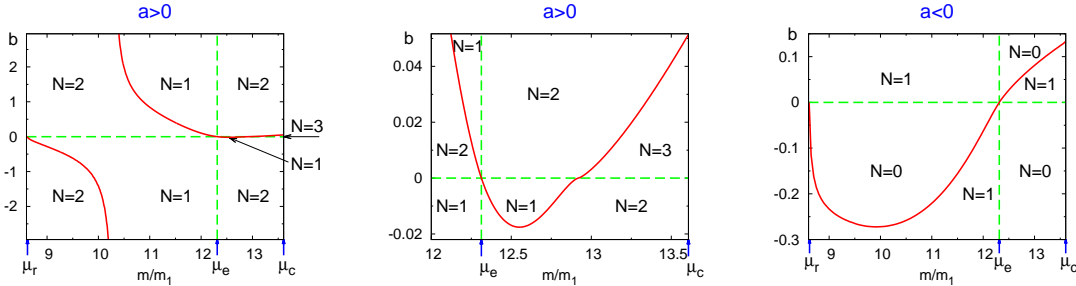


FIG. 2. A number of bound states in each domain of the $m/m_1 - b$ plane. Solid (red) line: critical three-body parameter $b_c(m/m_1)$ corresponding to the bound-state energy at the threshold. Dashed (green) lines: domain boundaries determined by $m/m_1 = \mu_e$ and $b = 0$. Part of the left panel is plotted in the middle panel to discern details. Values μ_r , μ_e and μ_c correspond to $\gamma = 1$, $1/2$ and 0 .

of special interest, viz., one finds for $a > 0$ that $b_c = 0$ at $m/m_1 \approx 12.91742$, $b_c \rightarrow \pm\infty$ at $m/m_1 \approx 10.2948$, $b_c \approx 0.05166$ at $m/m_1 = \mu_c$, and $b_c(m/m_1)$ has a local minimum $b_c \approx -0.01754$ at $m/m_1 \approx 12.550$. Similarly, one finds for $a < 0$ that $b_c \approx 0.13620$ at $m/m_1 = \mu_c$, and $b_c(m/m_1)$ has a local minimum $b_c \approx -0.2501$ at $m/m_1 \approx 10.15$.

Until now, in a number of reliable investigations of three two-component fermions (for

$m/m_1 \leq \mu_c$) [9–12, 28] it was explicitly or implicitly assumed that only one particular form of the wave function near the triple-collision point is allowed, which in terms of this Letter means that the three-body parameter b was set to zero. Nonetheless, the problem of two linear-independent square-integrable solutions was mentioned in [9, 18, 19, 29]. The two-parameter variety of three-body problems was defined in [19] by introducing the logarithmic derivative of the wave function at small hyper-radius; the relation of the present results and those of [19] is discussed in [27]. A rigorous treatment of few two-component fermions with the contact two-body interactions and the construction of a self-adjoint Hamiltonian was discussed from the mathematical point of view in [21–24]. The approach of [23] was further exploited in the calculation of three-body bound states [30].

The transition from the infinite Efimov spectrum to the one-parameter spectrum described in this Letter under the variation of the mass ratio is a general scenario, which will appear in a number of problems. One should anticipate the same transition for any problem, whose essential properties are determined by the effective potential with the singular part $\sim x^{-2}$, if its strength depends on a parameter (similar to the mass ratio). Evident example of this kind is the problem of three two-species particles in any L^P sectors [11, 12, 28]. Similar to the case of the $L^P = 1^-$ sector, the three-body parameter should be introduced in the L^- sectors of odd L and in the L^+ sectors of even $L > 0$ if two identical particles are fermions and bosons, respectively. Also, this scenario will be realised for the three-body problem in the mixed dimensions [31, 32] or in the presence of spin-orbit interaction [33, 34].

In future studies it is natural to find m/m_1 and b dependences of the scattering cross sections, three-body resonances, and recombination rates. The disclosed dependence on the three-body parameter should be taken into account in many-body properties as well; promising examples are the four-body (3+1) [35] and (2+2) [36] problems. Furthermore, the three-body parameter will be important in the crossover problem [37], i. e., in the relation of solutions for m/m_1 below and above μ_c ; another interesting point is the crossover of the solutions for m/m_1 above and below μ_r .

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SUPPLEMENTAL MATERIAL: UNIVERSAL DESCRIPTION OF THREE TWO-COMPONENT FERMIONS

Mass-ratio dependences of γ and q

The smallest eigenvalue of the auxiliary problem on a hyper-sphere $\gamma^2 \equiv \gamma^2(0)$ and its derivative $q = \left[\frac{d\gamma^2(\rho)}{d\rho} \right]_{\rho=0}$ are shown in Fig. S1. Note that the two-body scattering length is taken as a length unit ($|a| = 1$) and $q < 0$ ($q > 0$) for $a > 0$ ($a < 0$).

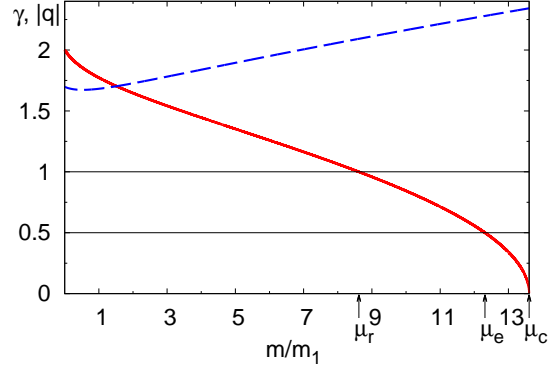


FIG. S1. The dependences of γ and $|q|$ on m/m_1 are depicted by the solid (red) and dashed (blue) lines, respectively. Values μ_r , μ_e , and μ_c correspond to $\gamma = 1$, $1/2$, and 0 .

Special mass-ratio values μ_r , μ_e , and μ_c

Few values of the mass ratio are of special interest, namely, μ_r , μ_e , and μ_c correspond to $\gamma = 1$, $1/2$, and 0 . Using Eq. (2) in the limit $\rho \rightarrow 0$ one comes to the equations

$$(\sin \omega_r + 1/2) \sin 2\omega_r - \omega_r = 0 , \quad (\text{S1})$$

$$\cos \omega_e \cos \frac{\omega_e}{2} - \frac{\sqrt{2}}{3} \tan \frac{\omega_e}{2} = 0 , \quad (\text{S2})$$

$$\frac{\pi}{2} \sin^2 \omega_c - \tan \omega_c + \omega_c = 0 . \quad (\text{S3})$$

Recall the definition $\sin \omega = 1/(1 + m_1/m)$. The roots of these equations are $\omega_r \approx 1.11075583$, $\omega_e \approx 1.18073571$, and $\omega_c \approx 1.19862376$ that correspond to the mass-ratio values $\mu_r \approx 8.61857692$, $\mu_e \approx 12.3130993$, and $\mu_c \approx 13.6069657$.

Three-body boundary conditions

It is suitable to write the three-body boundary conditions in the alternative form, viz., in terms of the derivative of the channel function $f(\rho)$. The boundary condition for $\mu_r < m/m_1 < \mu_c$ ($1 > \gamma > 0$), except $m/m_1 = \mu_e$ ($\gamma = 1/2$), which is equivalent to Eq. (3), reads

$$\lim_{\rho \rightarrow 0} \left(\rho^{1-2\gamma} \frac{d}{d\rho} \pm \frac{2\gamma}{|b|^{2\gamma}} \right) \frac{\rho^{\gamma-1/2}}{1-2\gamma+q\rho} f(\rho) = 0. \quad (\text{S4})$$

In the limit $m/m_1 \rightarrow \mu_c$ ($\gamma \rightarrow 0$) the boundary condition, which is equivalent to Eq. (4), takes the form

$$\lim_{\rho \rightarrow 0} \left(\rho \frac{d}{d\rho} - \frac{1}{\log(\rho/b)} \right) \rho^{-1/2} f(\rho) = 0, \quad (\text{S5})$$

where only $b > 0$ is allowed. In the specific case of $\gamma = 1/2$ ($m/m_1 = \mu_e$) the boundary condition

$$\lim_{\rho \rightarrow 0} \left(\frac{d}{d\rho} + \frac{1}{b} \right) \frac{f(\rho)}{1+q\rho \log \rho} = 0 \quad (\text{S6})$$

is equivalent to Eq. (5). Notice that the boundary condition for $\gamma = 0$ determined by Eq. (4) or Eq. (S5) is similar to that for the 2D zero-range model [26], whereas for $\gamma = 1/2$ the boundary condition of the form (5) or (S6) is similar to that for a sum of the zero-range and Coulomb potentials [38].

Solution for $m/m_1 = \mu_e$

A noticeable feature of the problem near $m/m_1 = \mu_e$ ($\gamma = 1/2$) is the degeneracy of energy dependences for different b and a lack of continuity in the definition of b . It is not surprising as the sign of the most singular term in HRE alters if γ goes across $1/2$. Due to discontinuity in the definition of b the limiting values of the bound-state energy for $m/m_1 \rightarrow \mu_e \mp 0$ ($\gamma \rightarrow 1/2 \pm 0$) do not coincide with each other and with that calculated exactly at $m/m_1 = \mu_e$ ($\gamma = 1/2$). The dependence of the bound-state energy on b is calculated using boundary condition (5) and plotted in Fig. S2. Notice that in boundary condition (5) one could substitute $\log \rho$ with $\log(\rho/\rho_0)$ introducing a scale ρ_0 , which simply leads to redefinition of length $\tilde{b} = b/(1 - b \log \rho_0)$.

The calculations for $a > 0$ show that there are two bound states, one of which disappears for $-0.108 < b \leq 0$; for $a < 0$ there is one bound state, which disappears for $-0.437 < b \leq 0$. In the limit $b \rightarrow \infty$ the bound-state energies tend to -4.319 and -1.061 for $a > 0$ and

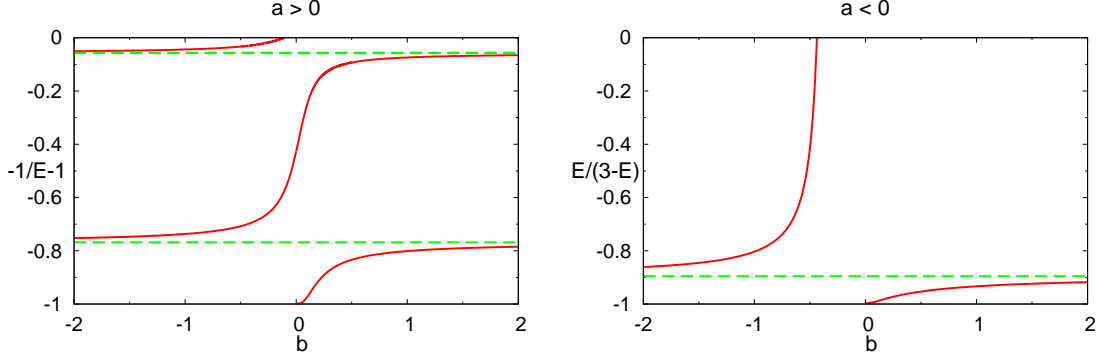


FIG. S2. Bound-state energies E as a function of b at $m/m_1 = \mu_e$ are plotted by solid (red) lines and asymptotic limits for $b \rightarrow \infty$ are indicated by dashed (green) lines. The two-body scattering length $a > 0$ (left) and $a < 0$ (right) and the energy axis scaled to map $-\infty < E < -1$ (left) and $-\infty < E < 0$ (right) to the interval $(-1, 0)$.

to -25.720 for $a < 0$. For $b = 0$ definitions (3) and (5) are the same and for $a > 0$ the bound-state energy takes the value $E_{00} \sim -1.74397$.

Zero-range limit of the three-body potential

Consider simple examples of the transition to the zero-range limit to clarify the introduction of the boundary condition at the triple-collision point $\rho \rightarrow 0$.

Square-well potential

Find the connection of the three-body parameter b and the parameters of the regularised potential defined as the square well $U(\rho) = -U_0$ for $\rho \leq \rho_0$ and as $U(\rho) = \frac{\gamma^2 - 1/4}{\rho^2} + \frac{q}{\rho}$ for $\rho > \rho_0$ in the zero-range limit $\rho_0 \rightarrow 0$. The solution of the equation $\left(\frac{d^2}{d\rho^2} - U(\rho) + E\right)f(\rho) = 0$ is $f(\rho) = \cos \kappa \rho$ ($\kappa = \sqrt{U_0 + E}$) for $\rho \leq \rho_0$ and is of the form (3) for $\rho > \rho_0$, which gives the relation

$$\kappa \rho_0 \tan \kappa \rho_0 = \gamma - \frac{1}{2} \pm 2\gamma \left(\frac{\rho_0}{|b|}\right)^{2\gamma} - \frac{q\rho_0}{1 - 2\gamma + q\rho_0}. \quad (\text{S7})$$

Up to the leading-order terms containing b and q , the potential strength U_0 is related to the interaction range ρ_0 as

$$U_0 = v \left[\frac{1}{\rho_0^2} \pm \frac{4\gamma}{|b|^{2\gamma}(\gamma^2 - 1/4 + v)\rho_0^{2(1-\gamma)}} + \frac{q}{(\gamma^2 - 1/4 + v)(\gamma - 1/2)\rho_0} \right], \quad (\text{S8})$$

where v is determined by $\sqrt{v} \tan \sqrt{v} = \gamma - 1/2$. Thus, the most singular term $\sim \rho_0^{-2}$ in the dependence $U_0(\rho_0)$ is determined by γ , whereas the parameter b determines less singular terms. With decreasing γ , the higher order terms containing b prevail over the term proportional to q , e. g., for $\gamma < 1/4$, the higher order term $\sim \rho_0^{-2+4\gamma}$ is more important than that of q/ρ_0 . For $\gamma = 0$ Eqs. (S7) and (S8) take the following form: $\kappa\rho_0 \tan \kappa\rho_0 = -\frac{1}{2} - \frac{1}{\log(\rho_0/b)}$ and $U_0 = \frac{v}{\rho_0^2} \left[1 + \frac{2}{(1/4 - v) \log(\rho_0/b)} \right]$. If $b = 0$, relation (S7) is not applicable; in this case the form (3) gives $\kappa\rho_0 \tan \kappa\rho_0 + \gamma + \frac{1}{2} = 0$ and $U_0 = \frac{\tilde{v}}{\rho_0^2}$, where $\sqrt{\tilde{v}} \tan \sqrt{\tilde{v}} = -\gamma - 1/2$.

Set-up of the logarithmic derivative

The wave function in the vicinity of the triple-collision point can be specified by imposing the three-body boundary condition for small ρ_0 , e. g., by setting the dimensionless logarithmic derivative of the channel function $\tan \delta = \rho \frac{d \log f}{d \rho}$ [19]. Using the asymptotic form of the solution as $\rho \rightarrow 0$ (3), one readily finds that for $\rho_0 \rightarrow 0$ two parameters δ and ρ_0 are related to the three-body parameter b as

$$|b|^{2\gamma} = \pm \rho_0^{2\gamma} \frac{\tan \delta - \gamma - 1/2}{[1 + q\rho_0/(1 - 2\gamma)] \tan \delta + \gamma - 1/2 + q\rho_0(\gamma - 3/2)/(1 - 2\gamma)}, \quad (\text{S9})$$

except for $\gamma = 1/2$. This relation could be used to link the results of [19] and those of the present Letter. For example, the dependence of the bound-state energy on δ in [19] is discontinuous at some δ_{cr} depending on ρ_0 . It stems from the discontinuous dependence of the parameter b on δ and ρ_0 ,

$$\tan \delta_{cr} = \frac{(1 - 2\gamma)^2 + q\rho_0(3 - 2\gamma)}{2(1 - 2\gamma + q\rho_0)}, \quad (\text{S10})$$

which follows from (S9) and for $\rho_0 \rightarrow 0$ takes the form $\tan \delta_{cr} = 1/2 - \gamma$, excluding the neighbourhood $\sim q\rho_0$ of the point $m/m_1 = \mu_e$ (of the order of $|\gamma - 1/2| < q\rho_0$). This exact expression can be compared with the dependence $\delta_{cr}(m/m_1)$, which was numerically calculated and presented in Fig. 5 of [19]. In particular, the exact expression gives that

$\delta_{cr} \rightarrow \arctan(1/2) \approx 0.46$ for $m/m_1 \rightarrow \mu_c$; the discrepancy with δ_{cr} in Fig. 5 of Ref. [19] indicates difficulty of the calculation in this mass-ratio limit.

Threshold solution

Critical dependence $b_c(m/m_1)$ for the appearance or disappearance of the bound state is determined by solving the eigenvalue problem for HREs at the two-body threshold $E = -1$ for the two-body scattering length $a > 0$ and at the three-body threshold $E = 0$ for $a < 0$. The square-integrability of solution follows from asymptotic behaviour of the HRE $\left[\frac{d^2}{d\rho^2} - U_{eff}(\rho) + E \right] f(\rho) = 0$, where $U_{eff} \rightarrow -1 + 2/\rho^2$ for $a > 0$ and $U_{eff} \rightarrow 35/(4\rho^2)$ for $a < 0$ as $\rho \rightarrow \infty$ [10]. Thus, if $\rho \rightarrow \infty$ the channel function $f(\rho)$ decays as ρ^{-1} for $a > 0$ and as $\rho^{-7/2}$ for $a < 0$. As usual, the bound state at the threshold turns to a narrow resonance under small variations of m/m_1 and b .